

# Natural Isomorphism from a Linear Map's Image to Complement of Nullspace

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## Abstract

There is a natural isomorphism from image to complement of nullspace, for a bounded linear map from a real Banach space onto a closed subspace of a real Hilbert space. This generalizes Riesz representation (self-duality of Hilbert space). The isomorphism helps solve the pressure equation of fluid dynamics.

Notation here holds throughout. A bound linear map  $A$ , from a real Banach space  $X$  to a real Hilbert space  $H$ , has closed image  $\mathcal{I}m(A)$ . (“Bound” means “bounded”.) The complement of nullspace  $\mathcal{N}(A)$ , or conullspace  $\mathcal{N}^\perp(A)$ , is  $\{f \in X^* : fx = 0 \text{ if } Ax = 0\}$ ;  $X^*$  is dual-space.  $A^t$  is transpose. Let  $h \in H$ ;  $(h|A)$  denotes the function  $\{X \ni x \mapsto (h|Ax)\}$  in  $\mathcal{N}^\perp(A)$ . The promised isomorphism is  $\tilde{A} : \mathcal{I}m(A) \ni h \mapsto (h|A) \in \mathcal{N}^\perp(A)$ .

**Note 1.**  $\tilde{A}$  is an isomorphism (of Banach spaces).  $\|\tilde{A}\| = \|A\|$ .

**Proof.** Clearly,  $\tilde{A}$  is linear. To show  $\tilde{A}$  is 1:1, we will show  $h = 0$ , if  $\tilde{A}h = 0$ . Since  $h \in \mathcal{I}m(A)$ , there is  $p \in X$  with  $Ap = h$ .  $0 = \tilde{A}h|_p = (h|Ap) = \|h\|^2$ ; hence  $h = 0$ , as promised.

Next, we will show  $\tilde{A}$  is onto. Let  $f \in \mathcal{N}^\perp(A)$ , seek  $h \in \mathcal{I}m(A)$ , with  $f = \tilde{A}h = (h|A)$ . Banach's closed-image theorem gives  $\mathcal{N}^\perp(A) = \mathcal{I}m(A^t)$ . So  $f = A^t \phi$ , for some  $\phi \in H^*$ ; Riesz-representation gives  $h_0 \in H$ , with  $(h_0|\cdot) = \phi$ . Also, because  $\mathcal{I}m(A)$  is a closed subspace, it gives a direct-sum:  $\mathcal{I}m(A) \oplus \mathcal{I}m^\perp(A) = H$ . Thus  $h_0 = h + h_\perp$  with  $h \in \mathcal{I}m(A)$ ,  $h_\perp \in \mathcal{I}m^\perp(A)$ . To verify  $f = (h|A)$ , let  $x \in X$ , and see:

$$fx = (A^t \phi)x = \phi(Ax) = (h_0|Ax) = (h + h_\perp|Ax) = (h|Ax).$$

Next, we will see  $\|\tilde{A}\| \leq \|A\|$ . If  $h \in \mathcal{I}m(A)$ , then

$$\|\tilde{A}h\| = \|(h|A)\| = \sup_{\|x\|=1} |(h|Ax)| \leq \|h\| \|A\|,$$

because  $|(h|Ax)| \leq \|h\| \|Ax\| \leq \|h\| \|A\| \|x\| = \|h\| \|A\|$ .

To conclude ( $\|\tilde{A}\| = \|A\|$ ), it remains to show  $\|\tilde{A}\| \geq \|A\|$ ; this will come by taking supremum (over  $p$ ) of the following bound:

$$\|\tilde{A}\| \geq \|Ap\| \quad \text{if } p \in X, \quad \|p\| = 1. \quad (1)$$

To prove (1), let  $p \in X$ ,  $\|p\| = 1$ ; then

$$\|\tilde{A}\| \|Ap\| \geq \|\tilde{A}(Ap)\| = \|(Ap|A)\| = \sup_{\|x\|=1} |(Ap|Ax)| \geq |(Ap|Ap)| = \|Ap\|^2. \quad (2)$$

If  $\|Ap\| = 0$ , then certainly  $\|\tilde{A}\| \geq \|Ap\|$ ; if  $\|Ap\| > 0$ , then divide (2) by  $\|Ap\|$ , to get (1). **Done.**

*Remarks.* • Scalars must be real. Try complex scalars, with scalar-product conjugate-linear in the first entry; then  $\{\tilde{A} : h \mapsto (h|A)\}$  is not linear.

• The Fredholm alternative  $\{\mathcal{I}m(A) = {}^\perp \mathcal{N}(A^*), \mathcal{I}m(A^*) = \mathcal{N}^\perp(A)\}$ , gives isomorphism  $\{\tilde{A} : \mathcal{I}m(A) \rightarrow \mathcal{N}^\perp(A)\}$  a dual expression:  $\{\tilde{A} : {}^\perp \mathcal{N}(A^*) \rightarrow \mathcal{I}m(A^*)\}$ , conullspace maps to image, for the transpose map. Here, of course,  ${}^\perp \mathcal{N}(A^*) = \{p \in H : (p|h) = 0 \text{ if } (h|A) = 0\}$ .

•  $\tilde{A} = A^* J|_{\mathcal{I}m(A)}$ , where  $J$  is the duality map:  $H \ni h \mapsto (h|\cdot) \in H^*$ .

• The isomorphism (image to conullspace) includes well-known facts. Riesz representation ( $H$  isomorphic to  $H^*$ ) is the case where  $X = H$  and  $A = I$  (identity). Also, since  $\mathcal{N}^\perp(A) = \mathcal{I}m(A^*)$ , the isomorphism links  $\mathcal{I}m(A)$  to  $\mathcal{I}m(A^*)$ : the images of a map and its transpose are naturally isomorphic; in particular, their dimensions match (row-rank equals column-rank, for matrices).  $\square$

**Acknowledgment.** *Note 1* builds on the “generalized Riesz theorem” of [Z], Chapter 5: if  $f \in \mathcal{N}^\perp(A)$ , then  $f = (h|A)$  for some  $h \in H$ . This follows from *Note 1*, since  $\tilde{A}$  is onto. (More is true:  $h$  may be chosen (uniquely) in  $\mathcal{I}m(A)$ , yielding an isomorphism.)  $\square$

Besides  $\tilde{A}$ , another natural isomorphism involving  $\mathcal{I}m(A)$  is the coset-map,  $\hat{A} : X/\mathcal{N}(A) \rightarrow \mathcal{I}m(A)$ ,  $\hat{A}(x+\mathcal{N}(A)) = Ax$ . Then  $\tilde{A}\hat{A}$  is a natural isomorphism from  $X/\mathcal{N}(A)$  to  $\mathcal{N}^\perp(A)$ .  $\tilde{A}\hat{A}(x+\mathcal{N}(A)) = (Ax|A)$ .

**Note 2.**  $\|\hat{A}\| = \|A\|$ ,  $\|\tilde{A}\hat{A}\| = \|A\|^2$ .

**Proof.** If  $\|x + \mathcal{N}(A)\| = 1$ , then for each  $\epsilon > 0$ , there is  $n \in \mathcal{N}(A)$  with  $1 \leq \|x + n\| \leq 1 + \epsilon$ . See

$$\|\hat{A}(x + \mathcal{N}(A))\| = \|A(x)\| = \|A(x + n)\| \leq \|A\| \|x + n\| \leq (1 + \epsilon) \|A\|.$$

Arbitrary smallness of  $\epsilon$  gives  $\|\hat{A}\| \leq \|A\|$ .

For each  $\epsilon > 0$ , there is  $x \in X$  with  $\|x\| = 1$  and  $0 \leq \|A\| - \|Ax\| \leq \epsilon$ . See

$$\|x + \mathcal{N}(A)\| \leq \|x\| = 1, \quad 0 \leq \|A\| - \|\hat{A}(x + \mathcal{N}(A))\| = \|A\| - \|Ax\| \leq \epsilon.$$

Arbitrary smallness of  $\epsilon$  gives  $\|\hat{A}\| = \|A\|$ . So also  $\|\tilde{A}\hat{A}\| \leq \|\tilde{A}\| \|\hat{A}\| \leq \|A\|^2$ ; and

$$0 \leq \|A\|^2 - \|\tilde{A}\hat{A}(x + \mathcal{N}(A))\| = \|A\|^2 - \|(Ax|A)\| \leq \|A\|^2 - |(Ax|Ax)| =$$

$$= \|A\|^2 - \|Ax\|^2 = (\|A\| + \|Ax\|)(\|A\| - \|Ax\|) \leq 2\|A\|\epsilon.$$

Arbitrary smallness of  $\epsilon$  gives  $\|\tilde{A}\hat{A}\| = \|A\|^2$ . **Done.**

*Remark.* The natural isomorphism,  $\tilde{A}\hat{A} : X/\mathcal{N}(A) \rightarrow \mathcal{N}^\perp(A)$ , generalizes the following familiar isomorphism, for the case where  $X = H$ , with closed subspace  $N$ , and  $A$  is the ortho-projector onto  $N^\perp$  (with  $\mathcal{N}(A) = N$ ). Because here  $A$  is self-adjoint with  $A^2 = A$ , here  $\tilde{A}\hat{A}$  identifies with the isomorphism  $X/N \ni x + N \mapsto Ax \in N^\perp$ .  $\square$

The isomorphism (image to conullspace) simplifies a standard argument in fluid dynamics, as in [G] (or [Z]). Consider flow in a region  $\Omega$ , a non-empty bound open connected subset of  $\mathbf{R}^3$ , with smooth border. As usual,  $\mathcal{L}_2(\Omega)$ ,  $W_2^1(\Omega)$ ,  $\dot{W}_2^1(\Omega)$  denote (respectively) the Lebesgue space of square-summable functions (modulo null measure, on  $\Omega$ ), the Sobolev space of functions with square-summable weak-rates, and its subspace of functions vanishing on the border.  $\{\mathcal{L}_2\}^3$  means  $\mathcal{L}_2(\Omega) \times \mathcal{L}_2(\Omega) \times \mathcal{L}_2(\Omega)$ , viewed as a Hilbert space of vector-fields on  $\Omega$ .  $\Gamma$  denotes the subspace of gradients;  $G \in \Gamma$  iff  $G = -\nabla p$  for some function  $p \in W_2^1$ ; in context of fluid dynamics, view  $G$  as density of external force, and  $p$  as pressure. Given a certain force-gradient  $G$ , we seek its pressure  $p$ ; to ensure uniqueness of pressure, require also  $\int_\Omega p = 0$ . The isomorphism (image to conullspace) will neatly solve the pressure equation  $\{-\nabla p = G, \int p = 0\}$ , and show continuity of the solution's dependence on data.

Let  $\nabla \cdot$  denote the divergence-map from  $\{\dot{W}_2^1\}^3$  to  $\mathcal{L}_2$ . Write  $\mathcal{L}_2^0 = \{f \in \mathcal{L}_2 : \int f = 0\}$ . The Divergence Theorem gives  $\mathcal{I}m(\nabla \cdot) \subseteq \mathcal{L}_2^0$ ; in fact,  $\mathcal{I}m(\nabla \cdot) = \mathcal{L}_2^0$ , as proven in [G].  $\mathcal{L}_2^0$  is closed (because it equals null-space of continuous function  $\{f \mapsto \int f\}$ ); hence  $\nabla \cdot$  has closed image, and *Note 1* applies (with  $A = \nabla \cdot$ ).

We will see  $\Gamma$  imbeds in  $\mathcal{N}^\perp(\nabla \cdot)$ , if we identify a gradient  $G$  with the function  $G^* = (G|\cdot)$ . Recall Helmholtz: each vector-field in the complement  $\Gamma^\perp$  equals the limit of a sequence of smooth vector-fields with divergence zero (“di-null”), vanishing near the region’s border (“border-null”). If  $V \in \mathcal{N}(\nabla \cdot)$ , then  $V$  is di-null and border-null, hence  $V \in \Gamma^\perp$ , and  $0 = (G|V) = G^* V$ ,  $G^* \in \mathcal{N}^\perp(\nabla \cdot)$ .

Since  $G^* \in \mathcal{N}^\perp(\nabla \cdot)$ , the inverse-isomorphism ( $\tilde{A}^{-1}$ ) of *Note 1* promises just-one  $p \in \mathcal{I}m(\nabla \cdot) = \mathcal{L}_2^0$ , with  $G^* = (p|\nabla \cdot)$ ; the map  $\{G \mapsto p\}$  is linear, continuous, and 1:1. It remains only to show  $p \in W_2^1$  and  $-\nabla p = G$ . Write  $G = (g_1, g_2, g_3)$ ; let  $\phi$  be any compactly-supported smooth function on  $\Omega$ ; denote by  $V$  the vector-field  $(\phi, 0, 0) \in \{\dot{W}_2^1\}^3$ . Then

$$\int p \partial_1 \phi = (p|\nabla \cdot V) = (G|V) = \int g_1 \phi,$$

which implies  $-\partial_1 p = g_1$ , weakly ( $W_2^1$ ). Likewise prove  $-\partial_2 p = g_2$ ,  $-\partial_3 p = g_3$ ;  $-\nabla p = G$ .

## References

- [G] Galdi, G. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, vol. 1. Springer, 1994.
- [Z] Zeidler, E. *Applied Functional Analysis: Main Principles and Their Applications*. Springer, 1995.